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Optimization of tree-shaped fluid networks with size limitations

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Abstract

In this paper, we show how to minimize the pumping power requirement in a fluid tree-shaped network under different size constraints (volume, surface, length). The Lagrange multiplier method is applied to obtain a problem formulation in which the pipe diameters do not appear explicitly. It is found that such a formulation exists for both volume and surface constrained networks. In Y-shaped junctions, optimal angles of branching and diameter ratios are determined. A different approach aiming at minimizing the network global cost (summation of size and pumping costs) is presented. It is showed that the geometrical features of the network are the same when one minimizes the global cost rather than minimizing pumping power under constraint. An optimal allocation of cost between pumping and size limitation was found. Finally, we extend the global cost minimization approach to the design of a porous architecture. This article provides fundamental tools for the designers of fluid tree-shaped networks.

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Keywords: Constructal; Fluid tree-shaped network; Designed porous media; Surface constraint; Pumping power minimization

1. Introduction

The principle-based generation of tree-shaped fluid networks (e.g., river basin, air passage in the lung, blood vascularisation, water distribution networks, solar panels, cold plate for the cooling of electronics) is a significant outcome of constructal theory [1–9]. The objective of fluid networks is to provide an easy access for fluid flow (low power dissipation). They are commonly used in energy transport and energy conversion systems (e.g., fuel cells [10,11], heat exchangers [12–15]). Optimizing such fluid networks means finding the best pipe connections (topology), diameters, mass flow rates, and branching angles for minimizing the power requirement or pressure drop. For optimal flow architecture to emerge, it is necessary to include global size constraints in the analysis (otherwise the result of the "optimization" is an infinitely large system) to take into account the fact that space is expansive and limited. Therefore, every part of the flow system must compete for filling the available space, and tradeoffs are to be made to allocate optimally the space while respecting the size-limiting requirements.

In the references mentioned above, Refs. [1–9], the pumping power requirement of fluid distribution networks is minimized based on a constrained total network volume. In several applications, however, important is not only the volume occupied by the network, but also its total surface area, its total length or a combination of different size-limiting constraints.

The main objective of this paper is to address the impacts of the network size-limiting constraints that are considered in the optimization (e.g., surface, volume, length), and of the way in which these size limitations are included in the analysis. First, we extend the equivalent network optimization problem formulation developed in Ref. [4] which is based on the Lagrange multiplier method in order to decrease the number of degrees of freedom (DOF) of the problem. This approach was originally developed for volume constrained networks, and is extended here to other size-limiting constraints (e.g., surface area). Next, we propose to include the size-limiting requirement directly in the function to be minimized: the size-limiting considerations are translated in terms of costs, and the total cost (summation of pumping power and size-related costs) is minimized. Finally, we design a pore network (i.e., porous medium) by taking into account the surface-related cost and the pumping power cost.

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Nomenclature

c	coefficient of proportionality	
C	cost	\$
D	diameter	m
f	friction factor	
h	Murray's law exponent	
K	permeability	m^2
L	length	m
$\dot{m}$	mass flow rate	$kg\,s^{-1}$
N	number of pipes	
S	surface	m^2
V	volume	m^3
$\dot{W}$	power requirement	W
Greek symbols		
α	exponent for the equivalent problem formulation	
ϕ	porosity	
λ	Lagrange multiplier	
ν	viscosity	$m^2\,s^{-1}$
θ	angle, degree	
ρ	density	$kg\,m^{-3}$
Subscript		
e, k	pipe (edge)	
min	minimized	
opt	optimal	
ref	reference	
Superscript		
$\sim$	dimensionless quantity	
$'$	quantity per unit of length	

The results presented in this paper are fundamental, and can be applied to any fluid flow tree-shaped network.

2. Laminar fully developed flow networks with surface area constraint

Consider a set of fluid users that are to be connected with pipes to a fluid source. The fluid consumption of every user is considered to be known, and so is their location. For example, a typical problem would be to connect a source located in the center of a disc-shaped area to users equidistantly positioned on the outer rim of the area [4,8], as shown in Fig. 2. The flow proceeds from the center of the disc to the outer rim. This problem became a benchmark for studying the generation of optimally shaped fluid tree networks [8]. The general problem of tree-shaped network optimization consists in finding the optimal topology (i.e., way of connecting the pipes) and pipe geometry (diameters, lengths) for minimizing power requirement under size constraint. Due to the potentially large number of degrees of freedom involved in the network optimization problem, it is in general a very challenging problem.

Assuming laminar fully developed flow in a pipe of length L and diameter D, the power dissipated in friction in the pipe is $128\nu\dot{m}^2L/(\pi\rho D^4)$ [4], where $\dot{m}$ is the mass flow rate in the pipe. The total power requirement for driving the flow in the entire network is simply the summation of the power dissipated in each pipe of the network, leading to

$$\dot{W} = \frac{128\nu}{\pi\rho}\sum_e \frac{\dot{m}_e^2 L_e}{D_e^4} \tag{1}$$

The summation in Eq. (1) is over all the pipes "e" of the network. To write Eq. (1) we neglected the junction losses, and we considered a Newtonian fluid with constant properties. Our objective is to minimize the power requirement, Eq. (1), by varying the network topology and the pipe diameters. Some authors prefer to minimize the total pressure drop rather than the pumping power. However, it has been demonstrated by Gosselin and Bejan [4] that in general minimizing pressure drop is not equivalent to minimizing pumping power. Therefore, we chose to minimize the pumping power in this paper.

Looking at Eq. (1), it becomes obvious that some size-limiting constraint involving the diameters of the pipes must be invoked in order to optimize the fluid network. Otherwise the diameters of the pipes in the optimal network would be infinitely large, $D_e \to \infty$, leading to $\dot{W} = 0$. This is clearly unrealistic. Heavy and large systems are expensive to build, maintain and transport, and this must be taken into account.

As mentioned in the introduction, a fixed total network volume V is often considered when optimizing fluid networks [1–9]. Here, we consider networks such that the size limitation is governed by a surface constraint. For example, in fluid distribution devices, the total pipe surface is a measure of the mass of the pipe system (surface × thickness of the wall × density) [16], which needs to be limited. Similarly, for the transport and distribution of hot water, the heat losses to the environment are proportional to the surface area [7]. Therefore these applications demand a small total surface area. The total surface area of the network is the summation of the surface area of each pipe,

$$S = \sum_e \pi D_e L_e \tag{2}$$

While minimizing Eq. (1) we want to respect the size constraint, Eq. (2). To do so, we apply the Lagrange multiplier method,

$$\frac{\partial \dot{W}}{\partial D_k} + \lambda\frac{\partial S}{\partial D_k} = 0, \quad k = 1, \ldots, N \tag{3}$$

where the index k runs from 1 to N, the number of pipes in the network. The combination of Eqs. (1)–(3) delivers the Lagrange multiplier λ

$$\lambda = \frac{512\nu\dot{m}_k^2}{\pi^2\rho D_k^5} \tag{4}$$

In more speaking words, we can say from Eq. (4) that the ratio $\dot{m}^2/D^5$ is a constant for all the pipes in an optimized network. Similarly, we note that in an optimal network, two

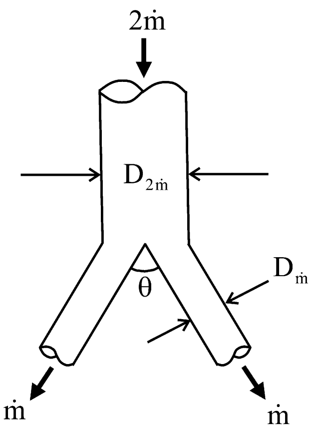

Fig. 1. The geometric features of a Y-shaped junction with equal splitting of the incoming flow.

pipes have the same diameter when they carry an equivalent mass flow rate. Eq. (4) is equivalent to Murray's law [17]. A Y-shaped junction with equal division of the flow, Fig. 1, is such that a larger mass flow rate $2\dot{m}$ divides into two smaller $\dot{m}$-streams. This means that the optimal diameter ratio between the two levels of branching in Fig. 1 (i.e., between the $2\dot{m}$- and the $\dot{m}$-pipes) for a network with a fixed surface area is

$$\left(\frac{D_{\dot{m}}}{D_{2\dot{m}}}\right)_{\text{opt}} = 2^{-2/5} \tag{5}$$

This result is different than in volume constrained networks where this ratio was $2^{-1/3}$ [4,8,17]. In other words, the pipe diameters in optimal V-constrained networks will tend to look more uniform than in S-constrained networks. Eq. (5) has been reported earlier for a surface constrained T-shaped duct assembly by Bejan et al. [16].

Introducing the Lagrange multiplier, Eq. (4), in Eq. (1) we obtain

$$\dot{W} = \frac{\lambda}{4}\sum_e \pi D_e L_e = \frac{\lambda}{4}S \tag{6}$$

The Lagrange multiplier λ is proportional to the power requirement. In other words, minimizing λ is equivalent to minimizing $\dot{W}$. Eq. (4) is then introduced into the constraint, Eq. (2),

$$\frac{S\rho^{1/5}\lambda^{1/5}}{\pi^{3/5}2^{9/5}\nu^{1/5}} = \sum_e \dot{m}_e^{2/5} L_e \tag{7}$$

which allows us to write the power requirement (to be minimized with respect to the topology) as

$$\dot{W} = \frac{\pi^3 2^9 \nu}{4\rho S^4}\left(\sum_e \dot{m}_e^{2/5} L_e\right)^5 \tag{8}$$

Eq. (8) leads to two important conclusions. The first is that minimizing power requirement in a tree network when the total surface area is constrained (fixed value of S) is equivalent to minimizing the summation $\sum_e \dot{m}_e^{2/5} L_e$ (i.e., the Lagrange multiplier). This formulation has the significant advantage that it does not involve the pipe diameters (less degrees of freedom). The only unknown becomes the topology. The network can be optimized based on the sole mass flow rates and pipe lengths information, and the optimal diameters are deduced afterward with Eq. (4).

Note that the power requirement scales as S^{-4}: increasing the total surface area S results in smaller pumping power requirement. Larger networks are cheaper to use in terms of pumping power. In other words, the formulation proposed in this section is valid when one needs to limit or restrain the total surface. Otherwise, it would make more sense to increase S indefinitely for decreasing $\dot{W}$. We will discuss in Section 8 another situation where the total surface is to be increased as much as possible (e.g., solid–fluid reactors).

The second important conclusion is the possibility of calculating an optimal angle of branching in Y-shaped junction with equal splitting of the flow, like the one shown in Fig. 1. The quantity $\dot{m}_e^{2/5}$ can be seen as a cost per unit length (since the cost associated with each pipe in the summation of Eq. (8) is $\dot{m}_e^{2/5} L_e$), and based on that information, the optimal angle of branching in Fig. 1, under surface area constraint, can be deduced as described by Gilbert [18], and we obtain:

$$\theta_{\text{opt}} = 97.4378^\circ \tag{9}$$

The angle θ is indicated in Fig. 1. Note that this optimal angle of branching was 74.9346° in optimized volume-constrained networks [4,17]. A more general expression for optimal angles of branching in asymmetrical junctions is presented in Section 4.

3. Turbulent fully developed flow in smooth pipes with surface area constraint

In the previous section, laminar regime was assumed. In this section, we consider the optimization of turbulent flow networks with a surface constraint, Eq. (2). The pumping power requirement in a network with turbulent flow in smooth-surfaced pipes can be written as [4],

$$\dot{W} = \frac{1.1156\mu^{1/5}}{\pi^{4/5}\rho^2}\sum_e \frac{\dot{m}_e^{14/5} L_e}{D_e^{24/5}} \tag{10}$$

assuming fully developed flow of a Newtonian fluid with constant properties, and neglecting the losses at the junctions. Applying the Lagrange multiplier method with the surface constraint, just like we did in Section 2, leads to

$$\lambda = \frac{5.35\mu^{1/5}\dot{m}_k^{14/5}}{\pi^{9/5}\rho^2 D_k^{29/5}} \tag{11}$$

It can be shown that $\dot{W} = 5\lambda S/24$. In other words, the power requirement is still proportional to the Lagrange multiplier. Introducing Eq. (11) into the constraint, Eq. (2), we obtain

$$\dot{W} = \frac{3493.1\pi^4\mu^{1/5}}{\rho^2 S^{24/5}}\left(\sum_e \dot{m}_e^{14/29} L_e\right)^{29/5} \tag{12}$$

Eq. (12) means that minimizing $\dot{W}$ under global surface area constraint is equivalent to minimizing the summation $\sum_e \dot{m}_e^{14/29} L_e$, which does not involve the pipe diameters. It is only a topology-dependant quantity. In a Y-shaped junction with equal splitting like the one in Fig. 1, Eq. (11) leads to the following diameters ratio between two consecutive levels of branching [4]

$$\left(\frac{D_{\dot{m}}}{D_{2\dot{m}}}\right)_{\text{opt}} = 2^{-14/29} \tag{13}$$

The optimal angle of branching in such a junction is

$$\theta_{\text{opt}} = 91.3534° \tag{14}$$

In volume constrained turbulent networks (with smooth-surfaced pipes), the pipe diameters ratio was $2^{-7/17}$ and the optimal angle of branching was 55.5302° [4].

4. Turbulent fully developed flow in rough pipes with surface area constraint

When the surfaces of the pipes are rough, the friction factor f is almost independent on the Reynolds number (i.e., on $\dot{m}$ and D), and the pumping power reads as [4]

$$\dot{W} = \frac{32f}{\pi^2\rho^2}\sum_e \frac{\dot{m}_e^3 L_e}{D_e^5} \tag{15}$$

under the same assumptions described in Sections 2 and 3. Considering the fixed total surface area constraint, Eq. (2), the Lagrange multiplier becomes

$$\lambda = \frac{160 f \dot{m}_k^3}{\pi^3\rho^2 D_k^6} \tag{16}$$

It can be shown that the power minimization under surface area constraint with turbulent flow in rough pipes is equivalent to minimizing the quantity $\sum_e \dot{m}_e^{1/2} L_e$. The optimal diameter ratio and branching angle of a Y-shaped junction that split equally the mass flow rate, Fig. 1, are respectively,

$$\left(\frac{D_{\dot{m}}}{D_{2\dot{m}}}\right)_{\text{opt}} = 2^{-1/2}, \qquad \theta_{\text{opt}} = 90° \tag{17}$$

Note that the optimal diameter ratio above has been reported by Bejan et al. [16] for the case of a T-shaped pipe assembly. However, Eq. (17) is more general because it is valid for all types of equal splitting junctions.

5. General problem formulation and "organized" flow structure

Before presenting another way of including the size limitations in tree-shaped network optimization, we summarize in this section the main results related to the minimization of power requirement under surface constraint presented above. For the sake of comparison, we also report in this section the results of power requirement under volume constraint as described in Ref. [4].

One of the findings of Gosselin and Bejan [4] that can be extended to surface-constrained networks is the equivalence between minimizing power requirement, and the minimization of the quantity $\sum_e \dot{m}_e^{\alpha} L_e$, where the exponent α depends on the flow regime in the pipes, and the size constraint that is invoked (i.e., fixed volume or fixed surface). This summation is only topology dependant. The possible values of α are reported in Table 1. The values of α are larger when the flow is turbulent than when it is laminar. Furthermore, α is smaller when a surface area constraint is invoked than with the volume constraint.

Table 1
Exponent α for the formulation of the equivalent problem

Flow regime	V-constraint	S-constraint
Laminar	$2/3 \approx 0.6667$	$2/5 \approx 0.4$
Turbulent (smooth surface)	$14/17 \approx 0.8235$	$14/29 \approx 0.4828$
Turbulent (rough surface)	$6/7 \approx 0.8571$	$1/2 \approx 0.5$

Table 2
Optimal angle of branching θ for equal splitting Y-shaped junctions (Fig. 1)

Flow regime	V-constraint	S-constraint
Laminar	74.9346°	97.4378°
Turbulent (smooth surface)	55.5302°	91.3534°
Turbulent (rough surface)	50.1581°	90°

In Table 2, we reported the optimal angle of bifurcation in Y-shaped junctions in which the flow is equally split (see Fig. 1). To illustrate how this information could be used, we considered the benchmark problem described above and by Wechsatol et al. [8]: a set of N users are positioned equidistantly on the outer rim of a disc-shaped area, and we want to connect them to a source located in the center of the disc. The "simplest" configuration is to connect each user to the center with its own single pipe (zero level of branching). This is also the most expensive configuration in terms of power requirement. To reduce the power requirement, one can increase the complexity of the dendritic structure by introducing junctions or bifurcations (levels of branching). It has been shown that for a given number of users on the periphery (N) there is an optimal number of levels of branching [8]. The optimal position of the branching points depends on the flow regime considered and the invoked size constraint.

In Fig. 2, we reported the optimal dendrites (for different flow regimes and size constraints) with two levels of branching when 16 users are positioned equidistantly on the outer rim of the disc. In a given optimized dendrite ((a), (b), (c) or (d)), the angle between the outlet branches of the Y-shaped junctions (see Fig. 1) is a constant (see Table 2) as only junctions that split the flow equally are considered. This angle is labelled θ in Fig. 2. However, the actual value of the angle of branching depends on the flow regime and size constraint considered. This results in very different looking dendrites: the branching points are closer to the center in Fig. 2(b) (optimal network with turbulent regime (rough surface) and volume constraint) and closer to the outer rim in Fig. 2(c) (optimal network with laminar regime with surface constraint). Note that in earlier work on dendritic flow architectures [8], the optimal angles of Table 2 have not been revealed in every junction because it was assumed that the lengths of the pipes at a given level of branching were equal, as discussed by Gosselin and Bejan [5].

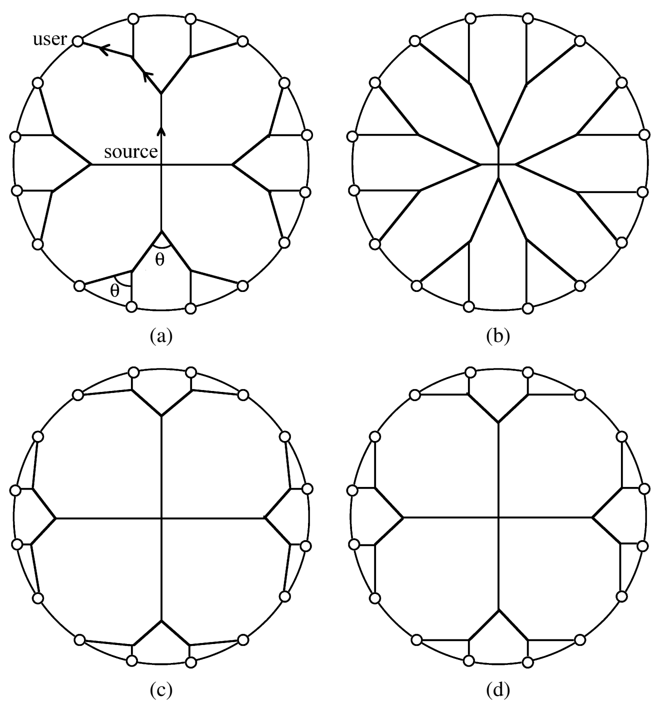

Fig. 2. The optimal dendrites with two levels of branching for different flow regimes and size constraints: (a) laminar regime with volume constraint, (b) turbulent regime (rough surface) with volume constraint, (c) laminar regime with surface constraint and (d) turbulent regime (rough surface) with surface constraint.

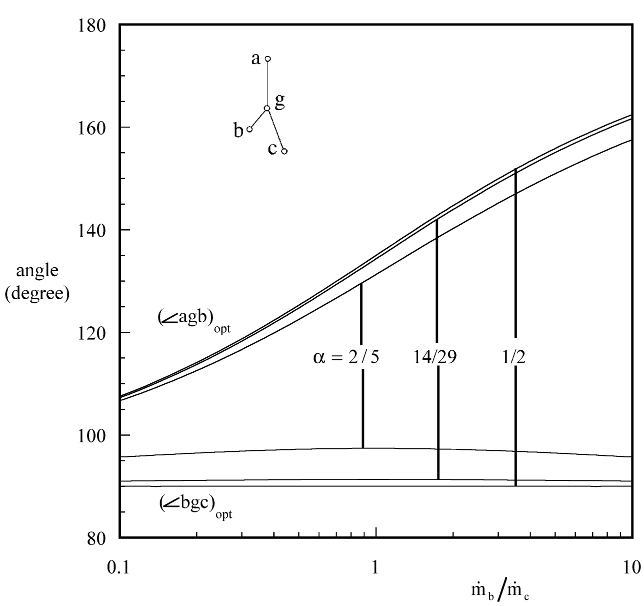

Fig. 3. The optimal angles around a junction point, Eqs. (18)–(20), with surface constraint.

Table 3
Optimal exponent h for the diameter ratio between two consecutive levels of branching (Murray's law), Eq. (21)

Flow regime	V-constraint	S-constraint
Laminar	$1/3 \approx 0.3333$	$2/5 = 0.4$
Turbulent (smooth surface)	$7/17 \approx 0.4118$	$14/29 \approx 0.4828$
Turbulent (rough surface)	$3/7 \approx 0.4286$	$1/2 = 0.5$

An asymmetrical Y-shaped junction is shown in Fig. 3. The fluid flow goes from the point "a" to the branching point "g" at a mass flow rate $(\dot{m}_b + \dot{m}_c)$. Then, a part of the flow $(\dot{m}_b)$ is deviated to the point "b", and the remaining $(\dot{m}_c)$ is transported to "c". In such a Y-shaped junctions the optimal angles $\angle bgc$, $\angle agc$ and $\angle agb$ can be determined straightforwardly [18] by

$$\cos(\angle bgc) = \frac{(\dot{m}_b + \dot{m}_c)^{2\alpha} - \dot{m}_b^{2\alpha} - \dot{m}_c^{2\alpha}}{2\dot{m}_b^{\alpha}\dot{m}_c^{\alpha}} \tag{18}$$

$$\cos(\angle agb) = \frac{\dot{m}_c^{2\alpha} - (\dot{m}_b + \dot{m}_c)^{2\alpha} - \dot{m}_b^{2\alpha}}{2\dot{m}_b^{\alpha}(\dot{m}_b + \dot{m}_c)^{\alpha}} \tag{19}$$

$$\cos(\angle bgc) = \frac{\dot{m}_b^{2\alpha} - (\dot{m}_b + \dot{m}_c)^{2\alpha} - \dot{m}_c^{2\alpha}}{2(\dot{m}_b + \dot{m}_c)^{\alpha}\dot{m}_c^{\alpha}} \tag{20}$$

The value of the exponent α is presented in Table 1, and depends on the flow regime, and size constraint. Note that the quantities of the type $\dot{m}^{\alpha}$ in Eqs. (18)–(20) represent the cost per unit length of the pipe carrying the flow rate $\dot{m}$. The angles of Eqs. (18)–(20) are reported in Fig. 3 as a function of the parameter $\dot{m}_b/\dot{m}_c$ for the values of α corresponding to surface area constrained networks. Important is that the angle $\angle bgc$ between the two outlet branches of a junction is almost constant irrespectively to $\dot{m}_b/\dot{m}_c$. An equivalent figure for optimized volume constrained networks was proposed by Gosselin and Bejan [4]. When $\alpha = 1/2$ (turbulent flow in rough pipes with a surface constraint), the angle is exactly equal to 90°, independently of the value of $\dot{m}_b/\dot{m}_c$. For optimizing a fluid network, Fig. 3 can be used straightforwardly for determining the optimal angles of branching (or equivalently, the position of the branching point). Note further that when $\dot{m}_b/\dot{m}_c = 1$, we obtain the results of Table 2 for equal splitting junctions.

A very general and fundamental result is the relation between the pipe diameters and mass flow rates in both volume and surface constrained networks (Murray's law), which can be written as

$$\left(\frac{D_{\dot{m}_1}}{D_{\dot{m}_2}}\right)_{\text{opt}} = \left(\frac{\dot{m}_1}{\dot{m}_2}\right)^{h} \tag{21}$$

where the exponent h changes with the flow regime and size constraint. The possible values of h are reported in Table 3. The changes in diameters from one level of branching to another are less important when the surface constraint is invoked than with the volume constraint. In human lung [20], the diameter ratio between two successive levels of branching is a constant close to the entrance (trachea). However, as the airways get closer to the alveoli, the changes in diameter between two consecutive levels of branching are smoother. In other words, the exponent h is smaller. This could be explained in part by the fact that close to the alveoli "surface" is more important or relevant than "volume".

It is important to mention that Eq. (21) and Table 3 are *results* of optimization, not *assumptions*. This feature distinguishes the constructal approach used in this paper from the fractal geometry. Fractal geometry does not predict the emergence of flow architectures, or the occurrence of governing relations like Eq. (21). It is descriptive in essence. Our approach predicts the relations between geometrical features of the network, or said differently, the occurrence of organized flow structures.

Before concluding this section, and considering another way of including the size limitation in the network optimization, it is worth to say a few words about the total network length. In some applications, the total length of the network may be a size-limiting requirement. For example, undergrounds networks require digging, the cost of which is directly related to the length of the network. The total length of a network is the summation of the length of each pipe,

$$L_{\text{tot}} = \sum_e L_e \tag{22}$$

Important to note is that minimizing power requirement (e.g., Eqs. (1), (10) or (15)), solely under total network length constraint is not possible. The total length of the network does not involve the pipe diameters, and therefore, the optimal diameters in such networks would tend to infinity in order to minimize $\dot{W}$. In other words, a constraint involving the diameters of the pipes is required to prevent the diameters to tend to infinity during the network optimization. An interesting feature of the proposed optimization formulation (i.e., minimize $\sum_e \dot{m}_e^\alpha L_e$) is that when one sets the exponent α equal to 0, the problem reduces to finding the shortest tree network (i.e., Steiner Tree Problem [19]), i.e. to minimize Eq. (22). The optimal angle of branching in Steiner trees is known to be 120° [19]. However, it should be remembered that optimizing the network on the sole basis of its total length is incorrect in the sense that it would result in infinitely large pipe diameters.

6. Minimizing global cost with surface-related cost

In the previous sections, we minimized power requirement under global size constraints (e.g., volume, surface). We discovered several geometrical features of the optimized networks under volume or surface constraint. An inconvenience of this approach (fixed volume V or surface S) is the difficulty to specify target values for V or S. In the end, the real concern is not really the total surface or volume of the network, it is the *cost* that this surface or volume represents. The costs associated with size limitations include for example the price of materials, manufacturing, transport, etc. Therefore, it makes sense to minimize the global cost of the network when optimize the network.

In this section, we show that the general problem formulation (minimize $\sum_e \dot{m}_e^\alpha L_e$) derived above when a fixed value of S or V was specified is also valid when one wants to minimize the global cost associated with the tree-shaped network. The global cost has two components: the cost of utilization (pumping power) and the cost related to the size limitation (e.g., building cost). For instance, let us consider a network with laminar flow and limited in size by the surface area constraint. The global cost C can be expressed as,

$$C = c_W \frac{128\nu}{\pi\rho} \sum_e \frac{\dot{m}_e^2 L_e}{D_e^4} + c_S \sum_e \pi D_e L_e \tag{23}$$

The coefficient c_W represents the cost of energy multiplied by the time of utilization of the network [$ W^{-1}] in such a way that the first term on the right-hand side of Eq. (23) is the cost associated with the pumping over a specified period of time (e.g., payment period). The second right-hand side term is an evaluation of the cost related to the size limitation, and the coefficient c_S is the cost per unit of surface area [$/m^2]. When the building cost is dominated by the cost of materials, c_S is the product of the pipes thickness and volumetric cost of the materials. For the transport of warm fluid, the coefficient c_S could characterize the cost associated with the heat losses to the ambient.

Calculating the derivative $\partial C/\partial D_k$, and setting it to zero leads to

$$D_k^5 = \frac{c_W}{c_S} \frac{512\nu}{\pi^2\rho} \dot{m}_k^2 \tag{24}$$

Introducing Eq. (24) in Eq. (23) yields the following cost function, minimized with respect to the pipe diameters,

$$C = \frac{\pi^{3/5} c_W^{1/5} c_S^{4/5} \nu^{1/5}}{2^{1/5} \rho^{1/5}} \sum_e (1+4) \dot{m}_e^{2/5} L_e \tag{25}$$

This expression is to be minimized with respect to the topology. Important in Eq. (25) is that the exponent to which the mass flow rate is raised is 2/5, the same that we found for surface-constrained networks with laminar flow, see Table 1. This means that the optimization results presented so far (i.e., optimal angles, diameters ratios) continue to apply, irrespectively of the actual values of c_W and c_S.

It is interesting to note that there is no equipartition of the total cost in this problem [21]. In the optimal laminar flow network limited in size by surface, the cost associated with the surface is 4 times as large as the one associated with the pumping, hence the factor $(1+4)$ in Eq. (25). Therefore, even though the total cost is not split equally between the two contributions, there is an optimal allocation of cost: 1/5 should go to pumping and 4/5 to the surface cost (e.g., materials cost). Again, this optimal allocation of the global cost does not depends on the values of c_W and c_S.

We repeated the procedure described above for turbulent flows and volume constrained networks. The optimization results of Tables 1–3 continue to apply for all the cases considered (i.e., laminar flow, turbulent flow, volume constraint, surface constraint, etc.) as the exponents to which the mass flow rates in relations similar to Eq. (25) are the same as in Table 1.

We present in Table 4 the optimal allocation of the global cost that is devoted to the pumping power. For example, we saw above that 1/5 of the total cost goes for pumping in an optimized surfaced-constraint network in laminar regime. The rest of the cost serves the cost associated with the size of the system (surface in the above case). When the size limitation cost

Table 4
Optimal cost allocated for the pumping power, $(C_{\text{pumping}}/C_{\text{total}})_{\text{opt}}$

Flow regime	V-dominated cost	S-dominated cost
Laminar	$1/3 \approx 0.3333$	$1/5 = 0.2$
Turbulent (smooth surface)	$5/17 \approx 0.2941$	$5/29 \approx 0.1724$
Turbulent (rough surface)	$2/7 \approx 0.2857$	$1/6 \approx 0.1667$

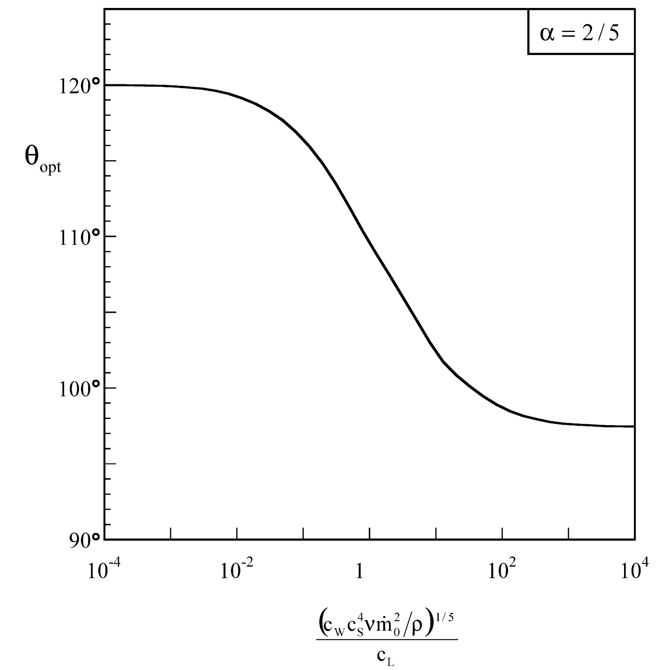

Fig. 4. The optimal angle of branching when the cost associated with the total length of the network is considered with the surface constraint.

is associated with the volume, a larger portion of the total cost should be devoted to pumping than when the cost is associated with the surface area. For example, in the laminar regime, 33% of the global cost goes for pumping when the limitation considered is V versus 20% when surface area S is the size limiting constraint. Turbulence decreases the cost allocation devoted to pumping.

The total network length, Eq. (22), can be included in the optimization problem by adding its contribution to the total cost function, yielding

$$C = c_W \frac{128\nu}{\pi\rho} \sum_e \frac{\dot{m}_e^2 L_e}{D_e^4} + c_S \sum_e \pi D_e L_e + c_L \sum_e L_e \qquad (26)$$

where c_L is a cost coefficient [\$/m] associated with the network length. Because L_{tot} does not involve the pipe diameters, we have $\partial L_{\text{tot}}/\partial D_k = 0$, and therefore Eq. (24) is still valid. The pipe diameter ratios presented in Sections 3–5 continue to apply. The total cost to minimize can be written as,

$$C = \sum_e L_e \left\{ \frac{5\pi^{3/5} c_W^{1/5} c_S^{4/5} \nu^{1/5}}{2^{1/5}\rho^{1/5}} \dot{m}_e^{2/5} + c_L \right\} \qquad (27)$$

The new cost per unit length is the expression in braces. In Fig. 3, we plotted the optimal angle between the outlet branches of a Y-shaped junction that split the flow equally, like the one in Fig. 1, as a function of the parameter $c_L^{-1}(c_W c_S^4 \nu \dot{m}_0^2/\rho)^{1/5}$. When $c_L \gg (c_W c_S^4 \nu \dot{m}_0^2/\rho)^{1/5}$, the cost per unit length is dominated by the cost associated with the pipe length. The problem then reduces to finding the shortest tree network (i.e., Steiner tree problem) leading to optimal angle of branching of 120°. On the other hand, when $c_L \ll (c_W c_S^4 \nu \dot{m}_0^2/\rho)^{1/5}$, the cost associated with the length of the network is negligible and the optimal angle of branching tends to the optimal value found in Section 4 (97.4378°).

7. Minimizing global cost with volume and surface limitations simultaneously

The next leap toward a more general formulation is to consider a total network cost that combines all the possible contributions: pumping power, surface cost, volume cost, and length cost,

$$C_{\text{tot}} = c_W \frac{128\nu}{\pi\rho} \sum_e \frac{\dot{m}_e^2 L_e}{D_e^4} + c_S \sum_e \pi D_e L_e + c_V \sum_e \frac{\pi}{4} D_e^2 L_e + c_L \sum_e L_e \qquad (28)$$

The derivative of C with respect to D_k leads to a polynomial expression of degree 6,

$$c_W \frac{512\nu \dot{m}_k^2}{\pi^2\rho} = c_S D_k^5 + \frac{c_V D_k^6}{2} \qquad (29)$$

for which an analytical expression of the form $D_k = D_k(\dot{m}_k)$ cannot be found. Note that the second derivative of C with respect to D_k is positive, which confirms the existence of a minimal cost. The non-existence of an analytical expression does not change conceptually the method outlined in this paper. For every mass flow rate, there is only one optimal pipe diameter, and we can optimize the network without pipe diameters, because the relation between D_k and $\dot{m}_k$, Eq. (29), is univocal.

To illustrate how Eq. (29) can be used, we considered the problem of finding the optimal angle of branching in an equal splitting Y-shaped junction. For the sake of simplicity, we neglect the cost associated with the network length, $c_L \to 0$. Eqs. (28)–(29) were re-written in the following way,

$$\frac{C_{\text{tot}}}{128\nu c_W \widetilde{m}_{\text{ref}}^2/(\pi\rho L_{\text{ref}}^3)} = \sum_e \tilde{L}_e \left(\frac{\widetilde{m}_e^2}{\widetilde{D}_e^4} + \gamma_S \widetilde{D}_e + \gamma_V \widetilde{D}_e^2 \right) \qquad (30)$$

$$4\widetilde{m}_k^2 = \gamma_S \widetilde{D}_k^5 + \frac{\gamma_V \widetilde{D}_k^6}{2} \qquad (31)$$

where the dimensionless mass flow rate and diameter are respectively $\widetilde{m}_k = \dot{m}/\dot{m}_{\text{ref}}$ and $\widetilde{D}_k = D/L_{\text{ref}}$, and

$$\gamma_S = \frac{\pi^2 \rho c_S L_{\text{ref}}^5}{128\nu c_W \widetilde{m}_{\text{ref}}^2} \quad \text{and} \quad \gamma_V = \frac{\pi^2 \rho c_V L_{\text{ref}}^6}{512\nu c_W \widetilde{m}_{\text{ref}}^2}$$

The mass flow rate and length scale used to non-dimensionalize $\dot{m}$ and D can be any constant scale of interest for a given problem. The term in parenthesis on the right-hand side of Eq. (30) represent the cost per unit length that is required to obtain the optimal angle of branching, see Section 4. Combining Eqs. (30)–(31), the cost per unit length can be expressed as a function of the mass flow rate only. We reported the pipe diameter and cost per unit of length ($\tilde{c}' = \widetilde{m}_e^2/\widetilde{D}_e^4 + \gamma_S \widetilde{D}_e + \gamma_V \widetilde{D}_e^2$) as a function of the mass flow rate in Fig. 5, for given values of γ_S and γ_V. This cost per unit of length was then used to calculate the optimal angle of branching presented in Fig. 6 [18]. The cost associated with the surface is proportional to $\widetilde{D}$, while the cost associated with the volume is proportional to $\widetilde{D}^2$. As the mass flow rate increases, the optimal diameter of the pipe increases as well, in such a way that for small mass flow rate (small $\widetilde{D}$) the dominating size-limiting contribution to the global cost comes from the surface cost. When the mass flow rate increases (larger $\widetilde{D}$), it is the volume limitation that dominates the global cost. This might explain in part the reason why "surface" is more important than "volume" in the lung closer to the alveoli, i.e. where the mass flow rate is smaller, as described in Section 4.

Introducing Eq. (30) into Eq. (31), we find that the optimized cost associated with the pumping power is the summation of one forth of the surface-related cost and one half the volume-related cost,

$$C_W = \frac{1}{4}C_S + \frac{1}{2}C_V \tag{32}$$

In other words, an optimized fluid tree-shaped network is such that it distributes optimally its expenses. This optimal allocation of cost can be a helpful tool for designing optimal networks.

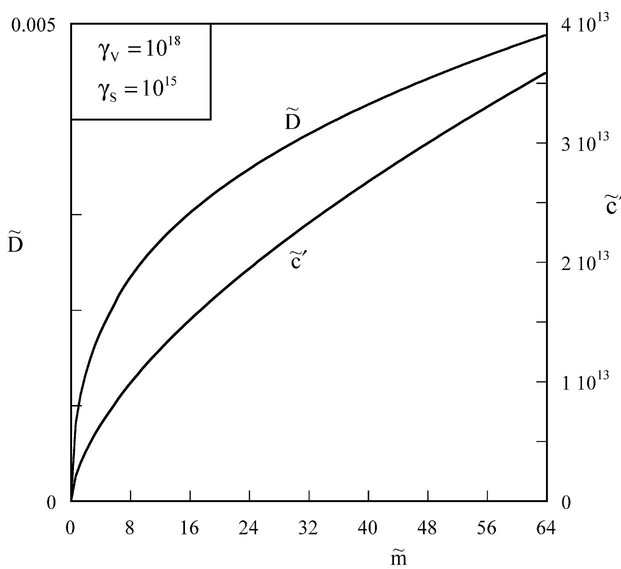

Fig. 5. The diameter and cost per unit length as a function of the mass flow rate when the global cost (pumping, volume and surface costs) is minimized.

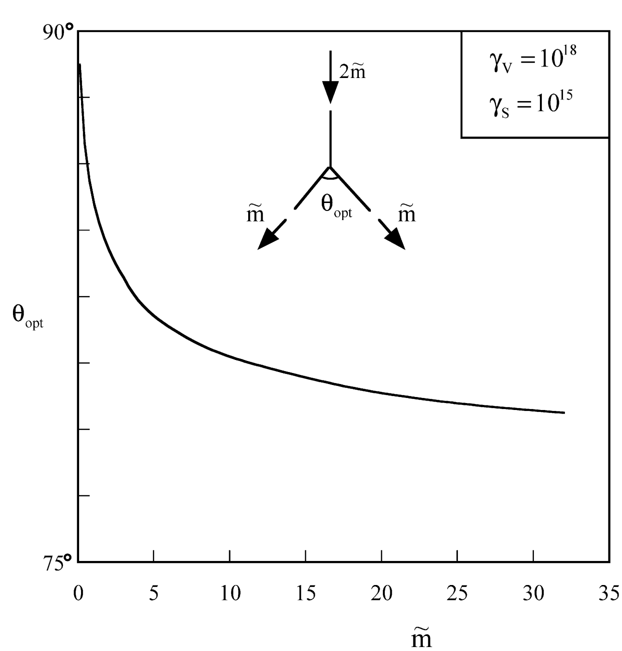

Fig. 6. The optimal angle of branching obtained by minimizing the global cost (pumping, volume and surface costs).

8. Surface-constrained designed porous media

In the previous sections, we wanted to limit or restrain the network total surface area. However, several energy and thermal engineering technologies rely on fluid–solid reactors [13,22] for which it is beneficial to increase the total surface area of the fluid–solid interface to maximize the density of chemical reactions rate. In heat exchangers, the number of transfer units (NTU) and the efficiency are related to the surface of exchange between the hot and cold streams [15]. For these systems, large surface area is required to maximize the density of the function served (e.g., heat transfer, chemical reaction).

In this section, we continue to tackle the same optimization problem with a different point-of-view. Consider a stacking of spheres through which a fluid flows, Fig. 7. The fluid and solid spheres system can be treated as a porous medium. The power requirement to drive the flow is

$$\dot{W}''' = \frac{(\dot{m}'')^2\mu}{\rho^2 K} \tag{33}$$

The power requirement of Eq. (33) is expressed per unit of volume. The permeability of a packing of spheres is related to the porosity and sphere diameter in the following way [2]

$$K = \frac{d^2\phi^3}{150(1-\phi)^2} \tag{34}$$

In the packing, as the spheres only touch at zero surface area contact points, the density of internal surface of contact (in m^2/m^3) between the fluid and the spheres is equal to the surface of one sphere time the number of spheres per unit of volume,

$$S''' = \frac{6(1-\phi)}{d} \tag{35}$$

We consider two contributions to the total cost of operation. The first contribution is proportional to the pumping power requirement, Eq. (33). The second contribution is related to the

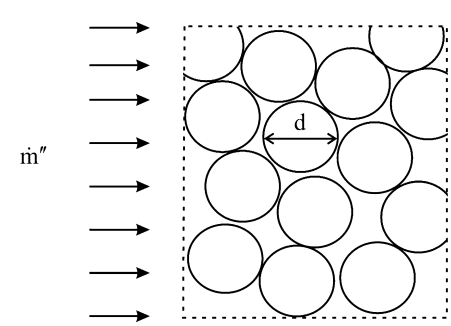

Fig. 7. The geometry of a solid–fluid reactor (porous medium).

density of surface area, Eq. (35). As mentioned above, in several fluid–solid reactors it is beneficial to increase the total surface area of the fluid–solid interface to maximize the density of chemical reactions rate. In other words, small surface area density is detrimental, and leads to larger cost in that case. Therefore, it makes sense to consider a surface related cost that is inversely proportional to S'''. The resulting total cost C_{tot} is

$$C_{\text{tot}} = c_{W'''} \frac{(\dot{m}'')^2 \mu (1-\phi)^2}{\rho^2 150 \phi^3 d^2} + c_{S'''} \frac{d}{6(1-\phi)} \tag{36}$$

where $c_{W'''}$ and $c_{S'''}$ are proportionality coefficients similar to the ones introduced in the previous sections. The minimization of Eq. (36) with respect to the diameter of the spheres yields

$$C_{\text{tot,min}} = 3.04 \frac{c_{W'''}^{1/3} c_{S'''}^{2/3} (\dot{m}'')^{2/3} \mu^{1/3}}{\rho^{2/3} \phi}$$
$$d_{\text{opt}} = 12.16 \frac{c_{W'''}^{1/3} (\dot{m}'')^{2/3} \mu^{1/3} (1-\phi)}{c_{S'''}^{1/3} \rho^{2/3} \phi} \tag{37}$$

In this problem, the optimal allocation of the total cost that goes for the pumping power is $1/3$. During the optimization, the volume of the pore network (or equivalently, the porosity, ϕ) was kept constant. The principle-based emergence of an optimal diameter for the solid particles of the porous medium stresses the idea of "designed porous media" [2,6].

9. Concluding remarks

In this paper, we presented the fundamentals of fluid tree-shaped network optimization under size constraints (i.e., surface, volume, length constraints). A size limitation must be included in the formulation of the network optimization problem for preventing unrealistic results to appear. An important outcome of this paper is the fact that size limitations can be included in the analysis either as a constraint to respect or directly into a global cost function to minimize.

For both volume and surface area limitations, the minimization of the power requirement is equivalent to minimizing $\sum_e \dot{m}_e^\alpha L_e$, where α depends on the constraint invoked, and the flow regime (Table 1). In other words, $\dot{m}_e^\alpha$ can be seen as a cost per unit length. This formulation does not involve the pipe diameters, and is only topology dependant. Again this formulation is valid both when the size limitation is considered as a constraint or in the global cost. Minimizing the global cost of a network leads to the same topology as minimizing the power requirement under surface or volume constraint.

The formulation presented in the preceding paragraph allowed us to find several geometrical rules that govern optimal networks. For example, in Y-shaped junctions with equal splitting of the flow, there exists an optimal branching angle that depends solely on the constraint invoked and the flow regime. Another example of these geometrical rules is the occurrence of an optimal diameter ratio between different levels of branching that depends solely on the constraint invoked and the flow regime, and based on the mass flow rate information.

The global cost minimization yielded the occurrence of an optimal allocation of the global cost between pumping and size limitation costs. In optimal network, the ratio of the global cost that should be devoted to pumping power is solely a function of the flow regime and dominating size constraint.

These conclusions are general and apply to any fluid network. Typical fluid network optimization problems involve large number of degrees of freedom (e.g., pipe diameters, mass flow rates, and topology) and are thus difficult to solve. Therefore, the results presented in this paper can be used for facilitating the design and analysis of fluid network systems.

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